XI. *On the Refraction of Plane Polarized Light at the Surface of a Uniaxal Crystal.*

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THE laws of the reflexion and refraction of polarized light at the surface of a crystal in accordance with the electro-magnetic theory of light have been discussed by LORENTZ (Schlömilch Zeitschrift, vol. xxii.), FITZGERALD (Phil. Trans., Vol. 171, 1880), and myself (Proc. Camb. Phil. Society, 1881). When a plane wave of electro-magnetic disturbance falls on the surface of separation between two different dielectric media six equations of condition are obtained. Three of these express the conditions that the electric displacement perpendicular to the surface and the electromotive force along the surface should be the same in the two media, while the other three do the same for the magnetic force and displacement. In all cases the six equations reduce to only four.

Let us suppose we know the amount and direction of the electric displacement in the incident wave. If both media are isotropic, these four equations give us the amounts and directions of the electric displacements in the reflected and refracted waves.

If the second medium is crystalline the possible directions of vibration in a wave travelling in it are known when the position of the wave is known; two of the equations as before give the amount and direction of the electric displacement in the reflected wave, the other two give the amounts of the displacements in the two refracted waves; the directions of these displacements being known from the position of the waves with reference to the axes of the crystal.

In general we have two refracted waves, the ordinary and extraordinary. Now, according to the electro-magnetic theory, light obeys the same laws as to propagation, reflexion, and refraction as this electro-magnetic disturbance, and the direction of the light vibrations coincides with that of the electric displacement, while the intensity of the light is measured by the energy of the disturbance. Our equations then give us the intensities of the two refracted rays which arise in general when a wave of polarized light falls on a crystal.

Consider now such an incident wave. Experiment tells us that there are two positions for its plane of polarization, in either of which one or other of the refracted waves disappears. The same result follows from the theory, and if we know the position of the face of incidence with reference to the axes of the crystal and also the directions of the two refracted wave normals, the theory enables us to calculate the angle between these two positions of the plane of polarization of the incident light.

But this angle is capable of direct measurement by experiment, and so the truth of the theoretical formulæ can be tested.

The experiment in its simplest form is as follows: a plane polarized beam of sodium light falls on a crystal of Iceland spar cut in the form of a prism, the position of whose faces relatively to the optic axis can be determined. The angle of incidence is observed and also the deviations of the two emergent wave normals.

From these data we can calculate the positions of the refracted wave normals in the crystal, and also of course the positions of the planes of polarization of the light travelling in these two directions respectively.

Let us suppose the polarizer to be a NICOL's prism, mounted in such a manner that the position of the plane of polarization of light emerging from it can be determined by means of a graduated circle attached to it; turn the NICOL until the extraordinary refracted wave disappears, then observe the position of the plane of polarization. But, there being only one ray, the ordinary, traversing the crystal prism, we can obtain from theory the azimuth of the plane of polarization of the incident light. Let us call this measured from some fixed plane θ_0. If the position of this fixed plane with reference to the NICOL can be found with accuracy we have here a means of comparing theory and experiment. We can eliminate the uncertainty in our knowledge of the relative position of the two planes of reference by turning the NICOL until the ordinary ray disappears, and reading the circle again; the difference between the two circle readings is the angle through which the plane of polarization has been turned; but the theory gives us again in this case, when there is only one refracted ray, the extraordinary, the value θ_E of the azimuth, measured from the same plane as before, of the plane of polarization of the incident light; the difference $\theta_0 \backsim \theta_E$ should be equal to the difference between the circle readings. Or again, having obtained as above a value for θ_0, alter the position of the spar prism so as to change the angle of incidence, and proceed as before; we can thus get a series of values of θ_0, the position of the plane of polarization of the incident light, for different angles of incidence as given by theory when the ordinary wave only traverses the crystal. But the readings of the polarizer circle give an experimental series of values of this same quantity and a comparison of these two series affords us a test of the theory. Since, in general, these two series of angles are not measured from the same zero point, there will, even if theory and experiment agree, be a constant difference between the two series depending on the difference of zeros. If the difference between corresponding values in the two series is not constant, but varies as the angle of incidence changes, we must infer that theory and experiment do not agree. We can test in the same manner the formula for the case in which the extraordinary wave only is propagated.

In practice, mainly in consequence of two difficulties, the experiments were conducted in a somewhat different manner. Unless certain rather elaborate conditions are fulfilled, the angle through which a NICOL'S prism is turned is not the angle through which the plane of polarization of the emergent light moves, and the difference between the two may, as I have shown (Phil. Mag., Nov., 1880, "On NICOL'S Prism"), be very considerable. Besides, when I began the experiments the only circle available was one graduated to 3′ of arc, which was not sufficiently accurate for my purpose. I allowed, therefore, plane polarized light to fall on my spar prism, and by turning it varied the angle of incidence until only one ray emerged. I observed the angle of incidence and the deviation, and from them calculated the azimuth of the plane of polarization of the incident light on the electro-magnetic theory. I then took a small cell with plane parallel glass sides and filled it with a weak solution of sugar; this I placed in the path of the incident light and thus produced a change in the position of its plane of polarization which brought the extraordinary ray into view again. This, by adjusting the angle of incidence on the spar prism, can again be made to vanish, and the angle of incidence and deviation being observed we can obtain a second value for the azimuth of the plane of polarization of the incident light. The angle between these two azimuths is the angle on the electro-magnetic theory through which the plane of polarization has been turned by the sugar cell. But this can be directly observed and the theory thus tested. Then I removed the sugar cell, altered somewhat the original plane of polarization, and again made the same observations, thus obtaining a series of values for the rotation produced by the cell corresponding to different angles of incidence on the spar prism.

The same observations were made using only the extraordinary ray. Thus the uncertainty arising from the want of adjustment and bad graduations of the polarizer circle was avoided.

The second difficulty was perhaps more serious. It was impossible to estimate with anything like sufficient accuracy the position of the spar prism for which the light of either ordinary or extraordinary ray was just quenched. To obviate this the apparatus was arranged as follows. The spar prism was mounted on the table of a spectrometer, kindly lent me by Professor STOKES, with a circle on silver and verniers reading to 10″. The instrument, and the method of adjusting the prism and focussing the telescope and collimator, have been described at length in my paper "On Plane Waves in a Biaxal Crystal" (Phil. Trans, 1879, p. 293). The sodium light was replaced by a strong source of white light, a powerful paraffin lamp, or the oxyhydrogen lime-light. A biquartz with the line of separation horizontal was placed between the polarizer and the collimator slit, and carefully adjusted by set screws, so that the light fell on it normally. Between the biquartz and the slit was placed a convex lens of about 20 centims. focal length, arranged so as to form an image of the biquartz on the slit of the collimator. The light from the slit fell on the spar prism, and two spectra, an ordinary and extraordinary, were formed and viewed by the telescope. Each of these spectra was divided horizontally into two parts, corresponding to the two parts of the biquartz, and when everything was adjusted the line of separation was seen clearly and distinctly.

Let us consider the ordinary image. Owing to the dispersion of the planes of polarization produced by the biquartz, the light of different colours in the incident wave is polarized in different planes, and a position can be found for the polarizer such that the plane of polarization of light of a certain colour on emerging from one-half of the biquartz is so related to the angle of incidence on the spar prism that the light of that colour is absent from the refracted beam. A dark band will be seen across one-half the spectrum in the place of this colour. Light of this colour emerging from the other half of the biquartz is in general not polarized in the same plane, and therefore in general, though a dark band will be formed in both halves of the spectrum, it will occupy different positions in the two. By turning the polarizer these bands appear to move in opposite directions across the field, and for one position of the polarizer the one can be brought vertically below the other. This position can be determined with great accuracy.

When this is the case the wave length of the light destroyed is clearly such that it has been rotated through 90° in opposite directions by the two halves of the biquartz. It is light then of a definite wave length, and we are thus able to place our prism with great accuracy in a position such that no light of one certain definite wave length is present in the ordinary wave. If, then, we are able to observe the angle of incidence and the deviation of the light of the same wave length in the extraordinary spectrum we shall have enough data to determine the azimuth of the plane of polarization of the incident light according to the electro-magnetic theory. It is easy enough to observe the angle of incidence. To find the deviation of the corresponding wave in the extraordinary spectrum, rotate the polarizer through about 90°; the dark bands will move out of the ordinary into the extraordinary spectrum, and the polarizer can be adjusted till they are brought to coincidence in it. When this is the case we know that it is light of the same wave length as before (viz.: that whose plane of polarization is turned through 90° by the biquartz), which is absent, and we have in the extraordinary spectrum a well-marked dark band, whose centre can easily be determined, and to which the needle point or cross wires of the observing telescope can be set with all the accuracy required. If we observe then the deviation of this dark band, we obtain the deviation in the extraordinary spectrum of the light which in the first part of our observation was wanting from the ordinary spectrum.

To escape the difficulty of having continually, when making observations for the determination of the position of the plane of polarization, to turn the polarizer through about 90° in order to get the deviation of the light in the extraordinary spectrum when the ordinary was quenched, or *vice versâ*, I divided the operation into two parts.

In the first I set the spar prism at a known angle of incidence and turned the polarizer until the dark bands coincided in the ordinary spectrum and then observed the deviation. I then turned the polarizer until the bands coincided in the extraordinary spectrum, and again observed the deviations; each of these observations was repeated five or six times and the mean taken. I then altered the angle of incidence by about 6° and made similar measurements.

In this manner I obtained a series of observations of angles of incidence ranging from 30° to 85°, with the corresponding deviations for both ordinary and extraordinary waves.

Let us call ϕ the angle of incidence, ϕ' that of refraction for the ordinary wave, ϕ'' for the extraordinary. From the observations the values of ϕ' and ϕ'' are easily determined by means of formulæ given by Professor STOKES (Brit. Ass. Report, 1862) and used by me in the paper already referred to.

If i be the angle of the prism ψ, ψ' the angles of emergence from and incidence on the second face, and D the deviation, we have

$$\psi = \mathrm{D} + i - \phi$$

$$\tan\frac{\phi'-\psi'}{2} = \tan\frac{i}{2}\tan\frac{\phi-\psi}{2}\cot\frac{\phi+\psi}{2}$$

$$\phi' + \psi' = i$$

and these equations give us ϕ'.

ϕ'' of course is found in the same manner.

I obtained thus a series of values of ϕ, ϕ', and ϕ''; now, of course, since ϕ' refers to the ordinary wave, if μ be the ordinary refractive index of the light used we should have $\frac{\sin\phi}{\sin\phi'} = \mu$, a constant, and this was found to be the case within small limits of error, showing that I had succeeded in quenching the same light throughout.

The value of μ was 1·662.

The values of $\phi'' - \phi'$ were also tabulated, and, of course, varied very slowly.

When this table had once been constructed, it was sufficient for the future to observe the angle of incidence; for knowing ϕ and μ, ϕ' is at once given by the formulæ $\sin\phi' = \sin\phi./\mu$ and ϕ'' by interpolation from the table. Thus the observations with the sugar-cell reduced to determining the angles of incidence at which the dark bands in the spectra coincided.

Each of these was determined five or six times and the mean taken.

We must now return to the theoretical considerations which enable us to express the azimuth of the plane of polarization of the incident light in terms of the angles of incidence and refraction and the position of the plane of polarization in the crystal.

Let us consider a plane-wave incident at an angle ϕ, let ϕ', ϕ'' be the angles of refraction for the two refracted waves respectively. The incident, reflected and refracted waves cut the plane face of incidence in the same line, let θ, $\theta_{,}$, θ' and θ'' be the angles between this line and the directions of the electric displacement in the incident reflected and refracted waves respectively.

Let a, $a_{,}$, a' and a'' be the amplitudes of the electric displacements in these directions.

Let q be the angle between the extraordinary wave normal and the corresponding ray.

Then (LORENTZ, 'Schlömilch,' xxii.; GLAZEBROOK, Proc. Camb. Phil. Soc., 1881) we have the equations

$$(a \cos \theta + a_{\prime} \cos \theta_{\prime}) \sin^2 \phi = a' \cos \theta' \sin^2 \phi' + a'' \cos \theta'' \sin^2 \phi'' \quad (1)$$

$$(a \sin \theta + a_{\prime} \sin \theta_{\prime}) \sin \phi = a' \sin \theta' \sin \phi' + a'' \sin \theta'' \sin \phi'' \quad (2)$$

$$(a \cos \theta - a_{\prime} \cos \theta_{\prime}) \sin \phi \cos \phi = a' \cos \theta' \sin \phi' \cos \phi' + a'' \cos \theta'' \sin \phi'' \cos \phi'' \quad (3)$$

$$(a \sin \theta - a_{\prime} \sin \theta_{\prime}) \sin^2 \phi \cos \phi = a' \sin \theta' \sin^2 \phi' \cos \phi' + a''(\sin \theta'' \cos \phi'' + \tan q \sin \phi'') \sin^2 \phi'' \quad (4)$$

These equations express the conditions referred to previously and enable us to find $a_{\prime}$ a' a'' and $\theta_{\prime}$; θ', θ'' are known from the position with reference to the axis of the wave front in the crystal.

We can solve them in the general case, but for our present purpose it is sufficient to find the conditions that only one wave should be propagated in the crystal. Let us first take the ordinary wave; we may put $a''=0$ in the equations, and we get the condition

$$\tan \theta = \tan \theta' \cos (\phi - \phi') \quad (5)$$

by eliminating $a_{\prime}$ a' and $\theta_{\prime}$.

This then is the condition which must hold between the position of the plane of polarization of the incident light and the angles of incidence and refraction in order that only the ordinary wave may traverse the crystal.

If we desire to have only the extraordinary wave, put $a'=0$ and we obtain

$$\tan \theta = \tan \theta'' \cos (\phi - \phi'') + \frac{\sin^2 \phi'' \tan q}{\cos \theta'' \sin (\phi + \phi'')} \quad (6)$$

In order to apply these formulæ we must find θ' θ'' and q in terms of the angles of incidence and refraction and constants.

Let the intersection of the incident, reflected and refracted waves with the face of incidence meet a sphere, centre O, in B (fig. 1).

Let the inward drawn normal to the face of incidence meet the same sphere in C, while the face itself cuts it in A B.

Let B I be the trace of the incident wave. B R of the refracted.

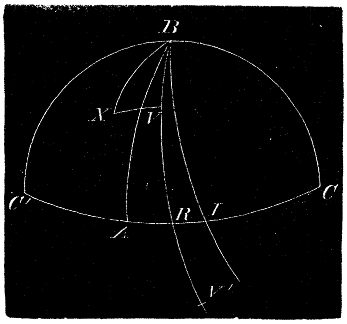
Let the optic axis cut the sphere in X. Join B X.

The prism used in the experiments was cut in such a way that X and R were on opposite sides of the arc B A, as in fig. 1.

Let B X$=\beta$ and let the angle A B X$=\lambda$. β and λ are known if the position of the face of incidence with reference to the optic axis is known.

Draw X V perpendicular to B R and take V V′ an arc of 90°. Then V V′$=\frac{\pi}{2}$; also O V, O V′ are clearly the two possible directions of vibration in the wave front B R. O V is the direction of the extraordinary vibration O V′ of the ordinary.

Fig. 1.



Let us suppose that B R is the ordinary refracted wave corresponding to an incident wave B I. Then A B V$=\phi'$ and X B V$=\lambda+\phi'$; also B V$=\frac{\pi}{2}-\theta'$; and from the right angle and triangle X B V we have

$$\cos \text{X B V}=\tan \text{B V} \cot \text{B X}$$

$$\cot \theta'=\tan \beta \cos (\lambda+\phi') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

But we have from (5)

$$\cot \theta=\cot \theta' \sec (\phi-\phi')$$

Therefore

$$\cot \theta=\tan \beta \cos (\lambda+\phi') \sec (\phi-\phi') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

If B R represent the extraordinary wave

$$\text{B V}=\theta'', \quad \text{A B V}=\phi'',$$

and we have

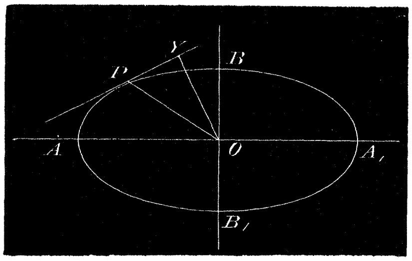
$$\tan \theta''=\tan \beta \cos (\lambda+\phi''). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

so that from (6) we obtain

$$\tan \theta=\tan \beta \cos (\lambda+\phi'') \cos (\phi-\phi'')+\sin^2 \phi'' \tan q / \cos \theta'' \sin (\phi+\phi'') \quad . \quad (10)$$

It remains now to find q, the angle between the extraordinary ray and the wave normal.

Fig. 2.



Let the figure (fig. 2) represent a section of the surface of wave slowness passing through the optic axis, O A, and the extraordinary wave normal, O P. Let O Y be perpendicular on the tangent plane at P, then O Y is, we know, the extraordinary ray. Let a and b be the principal refractive indices in direction O A and O B, and let $OP=r$, $OY=p$. Since OP and OY are respectively the directions of the wave normal and the ray, the angle $POY=q$, and $p=r\cos q$; let the angle $POA=\psi$.

Then

$$\frac{1}{p^2}=r^2\left\{\frac{\cos^2\psi}{a^4}+\frac{\sin^2\psi}{b^4}\right\}$$

Therefore

$$\sec^2 q=r^4\left\{\frac{\cos^2\psi}{a^4}+\frac{\sin^2\psi}{b^4}\right\}$$

Also

$$\frac{1}{r^2}=\frac{\cos^2\psi}{a^2}+\frac{\sin^2\psi}{b^2}$$

whence

$$\tan q=r^2\sin\psi\cos\psi\frac{a^2-b^2}{a^2b^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

But since r is a radius vector of the surface of wave slowness, $r=\sin\phi/\sin\phi''$, and we have

$$\sin^2\phi''\tan q=\frac{a^2-b^2}{a^2b^2}\sin^2\phi\sin\psi\cos\psi$$

Again X V (fig. 1) $=90^\circ-\psi$, and from the triangle X B V

$$\sin XV=\sin BX\sin XBV, \quad \text{or} \quad \cos\psi=\sin\beta\sin(\lambda+\phi'')$$

Also

$$\cos BX=\cos XV\cos BV, \quad \text{or} \quad \cos\beta=\sin\psi\cos\theta''$$

Thus

$$\sin^2\phi''\tan q=\frac{a^2-b^2}{2a^2b^2}\ \frac{\sin 2\beta\sin(\lambda+\phi'')\sin^2\phi}{\cos\theta''}$$

and equation (10) becomes

$$\tan\theta=\tan\beta\cos(\lambda+\phi'')\cos(\phi-\phi'')+\frac{a^2-b^2}{2a^2b^2}\ \frac{\sin 2\beta\sin(\lambda+\phi'')\sin^2\phi}{\sin(\phi+\phi'')\cos^2\theta''} \quad . \quad . \quad (12)$$

but θ'' can be found in terms of ϕ, ϕ'' and constants, from the formula

$$\tan\theta''=\tan\beta\cos(\lambda+\phi''),$$

and θ is thus expressed in terms of ϕ, ϕ'' and constants.

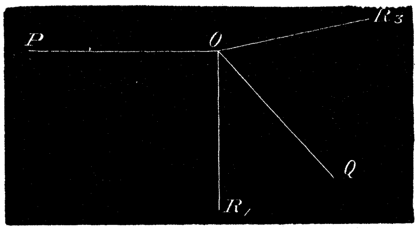
The value of a has been found already; it is the ordinary refractive index of the spar for the light used, and has been shown to be 1·662.

b is the extraordinary index, and may be taken with sufficient accuracy for the purpose from MASCART'S or RUDBERG'S determinations. Either of these give $b=1{\cdot}488$ as the value of the extraordinary index, corresponding to the value 1·662 for the ordinary.

In the spar prism used one face coincided very closely with a cleavage plane. The other face was inclined to this plane at an angle of 39° 17′ 20″, and the edge of the prism—that is, the intersection of the two faces—was nearly coincident with that of two cleavage planes.

Let O R_1, O R_2, O R_3 (fig. 3), be normal to the three cleavage planes; O P, O Q to the faces of the prism. The incident light fell on the face normal to O P. Then experiment proved that P, Q, R_3, R_1 were very closely indeed in the same zone; for the present we shall treat them as if they were accurately so, and this zone will therefore be the principal plane of the prism.

Fig. 3.



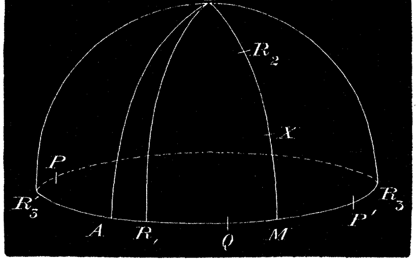
A series of observations on August 25, 1880, gave the values for the angles.

$$P\,O\,R_1 = 105°\ 54'$$
$$R_1\,O\,Q = 34°\ 48'$$
$$Q\,O\,R_3 = 40°\ \ 8'$$
$$P\,O\,R_3 = 180°\ 50'$$

Each observation was the mean of four or five closely concordant ones. The prism was reset and relevelled in November and another series of measurements taken, which agreed with the above to within 1′.

Let the optic axis and the edge of the prism meet the sphere in X and B respectively (fig. 4). Then B R_2 X is a great circle which bisects the arc R_1 R_3 in M say.

Fig. 4.



Let P O produced backwards meet the sphere in P′. Then

$$R_1R_2 = R_2R_3 = R_3R_1 = 74°\ 56'$$

and

$$XR_1=XR_3=XR_2$$

Also

$$\text{angle } R_1XM=60°$$

$$R_1M=\tfrac{1}{2}R_1R_3=37° \ 28'$$

whence

$$R_1X=44° \ 37'$$

and

$$XM=26° \ 15' \ 30'$$

Thus

$$BX=63° \ 44' \ 30'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and BX is the angle denoted by β in the formulæ.

Again we have

$$R_3P'=0° \ 50'$$

Therefore

$$MP'=36° \ 38'$$

In the arc $P'R_1$ take $P'A=90°$. Then B A is the trace of the face of incidence, and

$$\lambda=ABX=AM=90°-MP'=53° \ 22' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

These values of β and λ were used in reducing the experiments.

The error produced by assuming P, R_1, R_3, and Q to be in the same zone will be discussed later.

The results of the experiments are contained in the following tables.

Table I. is the interpolation table used as described above to find ϕ'' from the value of ϕ, and gives the series of corresponding values of ϕ, ϕ' and $\phi''-\phi'$.

Table II. gives the values of ϕ, ϕ' and θ for the case in which the ordinary wave only traversed the crystal.

ϕ is directly observed, ϕ' and θ are found from the formula

$$\sin \phi' = \sin \phi/\mu$$

and

$$\cot \theta = \tan \beta \cos (\lambda+\phi') \sec (\phi-\phi')$$

where $\mu=1{\cdot}662$, $\beta=63° \ 44' \ 30''$, $\lambda=53° \ 22'$.

The values are arranged in pairs. In the first experiment recorded in each pair there was no sugar cell in the path of the light. In the second experiment the sugar cell was interposed. The fifth column gives the differences between the two values of θ thus found, and this, if the formula given by theory were correct, ought to be the rotation of the plane of polarization produced by the sugar cell.

Each recorded value of ϕ is the mean of five observations. At high angles of incidence the difference between two observations of the same value of ϕ was sometimes as great as 10′. The error produced in the value of θ by an error $\delta\phi$ in ϕ is, for these values of ϕ, less than $\delta\phi/2$, so that the extreme difference between the values of θ calculated from each of the five values of ϕ, of which the mean is given in the table, may be as great as 5′. At high angles of incidence the mean error in the value of θ, as given in the table, is considerably less than 2′.

At lower angles of incidence, 45° to 30°, the differences in the observed values of ϕ were much less, and rarely amounted to 5′, but then the error produced in θ by an error $\delta\phi$ in ϕ is not far short of $\delta\phi$, so that the probable accuracy of the values of θ given in the table remains much the same as before.

TABLE I.

ϕ.	ϕ'.	$\phi''-\phi'$.
83° 11′ 0″	36° 41′ 10″	1° 2′ 40″
78 10 30	36 4 50	1 0 50
68 11 30	33 57 40	56 20
65 34 35	33 13 10	54 40
56 49 0	30 14 10	50 20
52 28 40	28 30 10	48 0
48 32 30	26 48 10	46 10
38 17 50	21 53 40	41 10
31 5 15	18 5 50	37 20
30 36 0	17 50 10	37 0
24 42 0	14 33 40	33 0
21 3 10	12 29 0	30 0
17 45 0	10 34 10	26 50

TABLE II.

	φ.	φ'.	θ−90°.	Rotation.
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
1	80 47 0 70 59 0	36 26 10 34 40 10	0 33 20 4 55 30	4 22 10
2	78 14 0 69 29 0	36 5 20 34 18 0	1 29 20 5 54 10	4 24 50
3	73 43 0 66 6 20	35 16 40 33 22 30	3 30 10 7 47·50	4 17 40
4	70 57 40 63 57 50	34 39 50 32 43 0	4 56 30 9 11 50	4 15 20
5	66 6 20 59 51 50	33 22 30 31 21 30	7 47 50 11 58 10	4 10 20
6	64 12 30 58 9 30	32 48 10 30 44 20	9 2 0 13 11 50	4 9 50
7	59 45 0 54 8 30	31 19 0 29 11 10	12 3 20 16 9 30	4 6 10
8	57 54 20 52 24 30	30 38 40 28 28 30	13 22 50 17 28 10	4 5 20
9	52 16 40 47 1 40	28 25 10 26 7 10	17 34 10 21 35 50	4 1 40
10	46 50 40 41 30 50	26 2 10 23 30 10	21 44 10 25 50 0	4 5 50
11	41 41 30 36 4 10	23 35 30 20 45 0	25 41 40 29 54 20	4 12 40
12	36 5 40 30 18 50	20 45 40 17 40 50	29 53 20 34 0 10	4 6 50
13	30 36 0 24 25 0	17 50 10 14 24 10	33 48 10 37 54 20	4 6 10
14	24 42 0 17 45 0	14 33 50 10 34 10	37 43 20 41 54 30	4 11 10

The rotation produced by the cell was measured carefully by FIZEAU's method, and the mean of several observations in which the extreme difference was about 5′ gave the value 4° 6′ 15″.

Each of the numbers then in Table II., column 5, ought to be equal to 4° 6′ 15″, and this is evidently not the case. The differences begin by being too large, and allowing for the probable error of the experiment, they decrease fairly uniformly as the angle of incidence decreases until we reach an angle of incidence of about 52°.

As the angle of incidence is still further decreased there is a tendency to increase in the values of the rotation given by the table. The value 4° 12′ 40″, line 11, is pretty clearly too big, as also possibly is that in line 14. A very small displacement in the position of the sugar cell, so that the light traversed it somewhat obliquely, would give rise to an error of the kind here considered. The cell was usually adjusted by observing the beam of light reflected from its first face. This could be made with a little care to travel back through the biquartz, and in that case the light clearly fell normally on the cell.

The first twelve sets of observations recorded in Table II. were made on October 28, 1880, the last two a few days later.

Thus, unless there is some regular source of error in the experiments, we must conclude that the formula connecting the plane of polarization and the angle of incidence in the case in which only the ordinary ray traverses a crystal of Iceland spar, as given by the electro-magnetic theory, is only true approximately.

The method does not enable us to determine accurately by experiment the position of the plane of polarization of the incident light with reference to the face of the crystal on which it falls; we can however compare the rate of change of the position of this plane, as the angle of incidence varies, found from experiment with its value deduced from theory.

We arrive at the conclusion that this rate of change as given by theory is too rapid when the angles of incidence are large, that for angles of incidence of from 30° to 60° the theoretical and experimental rates agree, while for lower angles of incidence the theoretical rate is again possibly too great; the last inference however being a little uncertain.

Two other shorter sets of observations made about the same time confirm these statements exactly, and it does not seem worth while to print a table of the numbers actually arrived at.

TABLE III.

	ϕ.	ϕ''.	θ.	Rotation.
	° ′ ″	° ′ ″	° ′ ″	° ′ ″
1	82 55 0 69 21 30	37 42 10 35 13 10	0 48 30 4 12 40	3 24 10
2	77 24 20 67 0 0	36 58 10 34 33 40	1 37 0 5 18 0	3 41 0
3	73 4 40 64 57 30	36 16 0 33 56 20	2 35 10 6 21 40	3 46 30
4	69 3 30 61 37 30	35 8 20 32 50 30	4 21 0 8 17 10	3 56 10
5	66 42 0 59 57 0	34 28 10 32 15 20	5 27 10 9 19 30	3 52 20
6	64 46 50 58 17 0	33 52 50 31 38 20	6 27 40 10 26 0	3 58 20
7	61 29 40 55 37 0	32 47 50 30 35 50	8 22 0 12 19 0	3 57 0
8	55 43 0 50 27 30	30 38 10 28 25 50	12 14 50 16 14 0	3 59 10
9	50 18 0 45 15 0	28 21 50 26 2 30	16 21 0 20 27 20	4 6 20
10	45 17 0 40 17 0	26 3 20 23 35 50	20 26 0 24 34 40	4 8 40
11	40 9 20 35 4 30	23 32 0 20 53 10	24 41 0 28 51 20	4 10 20
12	35 7 30 29 50 30	20 54 0 18 1 20	28 48 50 32 58 30	4 9 40
13	29 40 0 24 4 0	17 58 10 14 44 50	33 3 20 37 12 30	4 9 10
14	24 15 0 17 45 0	14 51 10 11 1 0	37 4 50 41 23 0	4 18 10

To complete the investigation we should consider the effects of possible errors in the constants β and λ. This can be done more advantageously after we have tabulated the results of the experiments in which the extraordinary ray only was allowed to traverse the crystal.

Table III. contains these. θ as before is the azimuth of the plane of polarization of the incident light, which in this case is calculated from the formula (12)

$$\tan\theta = \tan\beta\cos(\lambda+\phi'')\cos(\phi-\phi'')$$
$$+\frac{a^2-b^2}{a^2b^2}\,\frac{\sin 2\beta\sin(\lambda+\phi'')\sin^2\phi}{\sin(\phi+\phi'')\cos^2\theta''}$$

where

$$\tan \theta'' = \tan \beta \cos (\lambda + \phi'')$$

$$a = 1{\cdot}662 \qquad b = 1{\cdot}488$$

$$\beta = 63^\circ\ 44'\ 30'' \quad \lambda = 53^\circ\ 22'$$

The observations recorded were made on November 16, 1880.

For the rotation of the sugar cell, measured by FIZEAU'S method, the value 4° 4′ 20″ was obtained as the mean of seven measures, the two extremes of which differed by 5′.

If our theory then were correct, each of the differences given in Table III., column 5, should be 4° 4′ 20″.

We see at once that this is very far from being the case. The values of the rotation commence by being much less than 4° 4′ 20″ and, with the exception of Experiment 4, increase fairly regularly as the angle of incidence decreases.

As was the case with the ordinary ray, the rotations agree with experiment for an angle of incidence of about 50°; from that point onwards the theoretical rotation is too great.

Thus the theoretical rate of change in the position of the plane of polarization is too small for high angles of incidence, but increases as the angle of incidence decreases, and finally becomes too great.

Several other series of experiments lead to the same conclusions.

It remains then to discuss the effect of an error in the values of λ or β, and this is all the more necessary, for we know that the values taken are only closely approximate.

Let us take the ordinary wave first for which we have

$$\cot \theta = \tan \beta \cos (\lambda + \phi') \sec (\phi - \phi')$$

and consider the effect of decreasing β by a given amount.

The logarithm of cot θ is thus decreased throughout by the same quantity.

Now the change produced in θ by a given change in log cot θ is greatest when θ is nearest to 45°, thus by any change in β the values of θ will be more altered in Experiment 14 than in Experiment 1.

But we have to consider the change in the difference between two consecutive values of θ.

When θ is near 90° log cot θ changes more rapidly for a given change in θ than when it is near 45°.

If then in Experiment 1 we subtract from each of the values of log cot θ a certain quantity, it will alter the value of θ in the second line by an amount considerably greater than the alteration it produces in the first line, and thus the difference between the two values of θ will be reduced.

If on the other hand we consider Experiment 14, each of the values of θ there will be more affected by this change than in Experiment 1, but the effect produced on the value of θ in the first line will be nearly the same as that produced on the value of θ in the second, and thus the difference between the two values will not be so much altered.

Thus a decrease in β will decrease everywhere the theoretical value of the rotations, but it will affect the high angles of incidence considerably more than the low. It will thus tend to bring the theory more into accordance with experiments.

Let us see how it will affect the extraordinary ray.

The term in the formula with $\frac{a^2-b^2}{2a^2b^2}$ for a factor is practically a small and slowly varying correction except for the very highest angles of incidence; let us consider it as constant with regard to β and see how a decrease in β affects the values of θ supposed to depend only on the term

$$\tan\beta\cos(\lambda+\phi'')\cos(\phi-\phi'')$$

Exactly the same reasoning applies to this as in the case of the extraordinary wave.

The values of θ in Experiment 14 will be most decreased, but the alteration in the value of θ in the second line of Experiment 1 will be much greater than the alteration of the value of θ in the first line, while the change in the values of θ in Experiment 14 will be much the same for the two. Thus, by decreasing β the differences will be decreased throughout, but the changes will be greatest when the angles of incidence are large.

Thus a decrease in β will increase the differences between the theory and experiment.

Hence considering the ordinary and extraordinary rays together, a change in β will not reconcile the facts observed.

Calculation shows that decreasing β by 30′ decreases the differences in Experiments 4, 11, and 14 by 6′, 4′, and 1′ 20″ respectively for the extraordinary ray, and the amounts are about the same for the ordinary.

We must now consider alterations in λ; putting $\tan\beta=K$ we have for the ordinary ray

$$\cot\theta=K\sec(\phi-\phi')\cos(\lambda+\phi')$$

therefore

$$\delta\theta=K\sin^2\theta\sin(\lambda+\phi')\sec(\phi-\phi')\delta\lambda.$$

Now when ϕ is large θ is nearly 90° and $\phi-\phi'$ is large.

Thus $\sin\theta\sin(\lambda+\phi')$ and $\sec(\phi-\phi')$ all increase with ϕ', and therefore $\delta\theta/\delta\lambda$ is greatest when ϕ is large.

Thus the alteration produced in θ is greater the greater the angle of incidence.

If then we decrease λ we shall reduce in every case the differences between the corresponding values of θ, but we shall reduce them most for high angles of incidence and thus tend to bring the theory more into accordance with experiment.

Turning now to the extraordinary wave, and again considering the first term only, we find

$$\delta\theta = -K \cos^2\theta \cos(\phi-\phi'') \sin(\lambda+\phi'')\delta\lambda$$

Now as the angle of incidence decreases, $\cos^2\theta$ and $\sin(\lambda+\phi'')$ both decrease, but $\cos(\phi-\phi'')$ increases, and we must have recourse to numerical values most easily to consider the changes in the product.

Putting in the values for $\phi=82° 55'$ and $\phi=17° 45'$ we find $\delta\theta/\delta\lambda$ is numerically the greatest in the first case. Hence by decreasing λ we increase θ, and that most for the high angles of incidence.

Thus the decrease in λ increases the first value of θ in each of the experiments in Table III. by a greater quantity than the second. It therefore decreases the differences throughout, and does not tend to bring theory and experiment into agreement.

Hence taking both rays into account an alteration in λ will not produce the effect required.

Thus it is not possible to change the constants β and λ so as to produce greater agreement between theory and experiment.

But β and λ are both measured from the intersection of the incident wave and the face of incidence. In treating them as constants we have supposed that as the prism was turned this line of intersection remained parallel to itself, thus assuming that the prism was accurately levelled, so that the incident and refracted wave normals lie always exactly in a principal plane.

If this be not the case, β and λ become functions of the angle of incidence, and have not strictly the same values in the consecutive lines of Tables I. and II.

To test the effect of this error experimentally I set the spar prism so that the incident wave normal instead of lying in the principal plane was inclined to it at about 10′, and made a short series of measurements.

The results agreed very closely with those tabulated in Tables I. and II., and proved conclusively that the level error which I feel sure never amounted to as much as 2′ could not account for the difference between theory and experiment.

The same is fairly clear from the formulæ, for in Table II. we could reconcile theory and experiment by supposing that at high angles of incidence β is somewhat greater than 63° 44′ 30″, and that it decreases continually as the angle of incidence increases, but this assumption would just tend to increase the differences given in Table III.

Thus a small variable error in the value of β will not account for the observed facts.

One point, however, needs further investigation. An error has been confessedly committed in the value of β. Can we assign a limit to its value?

The fact that the four poles P, Q, R_1, R_3 are nearly but not quite in the same zone renders it impossible to determine accurately the position of P and Q. The prism was levelled by placing a mark across the slit, which when viewed directly appeared to coincide with the end of the needle point in the focus of the telescope. The telescope was then turned to view the images reflected from the faces of the prism, and the levelling screws of the prism adjusted until the same coincidence was established.

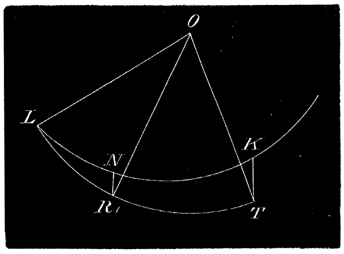
Now observation showed that when P and Q were level in this manner R_1 and R_3 were both a very little too low.

The vertical angular distance between the end of the needle and the mark across the slit, seen by reflection from R_1 or R_3, could be estimated by setting the needle on the slit, reading the vernier of the telescope and then moving the telescope with the needle until the horizontal distance between it and the slit appeared about equal to the vertical distance between the end of the needle and the mark.

This horizontal distance is found at once by again reading the vernier of the telescope.

I found as the mean of several closely concordant observations that when the angle of incidence of the light on the face R_1 was 20° 58′ the image formed by reflexion was 10′ too low, and the same exactly was the case with the image formed by reflexion at the same angle from R_3.

Fig. 5.



Let L O (fig. 5) be the direction of the incident light from the mark on the slit, O T the reflected ray, O R_1 the normal to R_1.

L R_1 T is a great circle, let L N K be the principal plane of the prism, T K and R_1 N being perpendicular to L N K.

$$LR_1=\phi=R_1T$$

Let the angle $TLK=\gamma$, $R_1N=x$.

The experiment has shown that $TK=10'$.

From the triangles T L K, R_1 L N

$$\sin x=\sin\gamma\sin\phi$$
$$\sin 10'=\sin\gamma\sin 2\phi$$

and

$$\phi=20°\ 58'$$

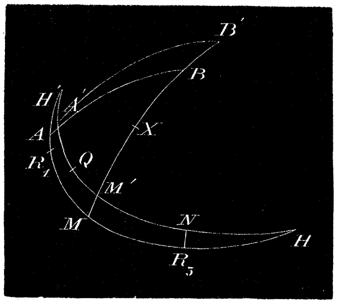
whence

$$x=6'$$

Thus R_1 is 6′ below the great circle L K, and so also is R_3.

Thus in reality the principal plane of the prism is inclined at a small angle to the plane R_1 R_3, the intersection of the two being the exterior bisector of the angle R_1 O R_3. Let H H′, fig. 6, be the line of intersection, B′, M′ the new position of B and M respectively, X the optic axis.

Fig. 6.



Then M, M′, X, B, B′, lie on a great circle, and BB′=MM′=MHM′

$$R_3H=90°-37° 28'$$
$$R_3N=6'$$
$$\sin MM'=\sin R_3N \operatorname{cosec} xR_3H$$

whence

$$MM'=8'$$

And if $\delta\beta$ be the change thus produced in β

$$\delta\beta=XB'-XB=BB'=8'$$

and if A′ is the new position of A

$$MA=\lambda \quad M'A'=\lambda+\delta\lambda$$

but

$$MA=M'A' \quad \therefore \delta\lambda=0$$

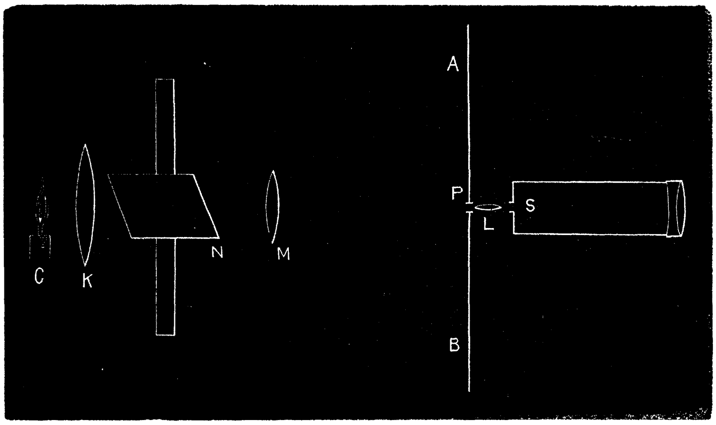
The value of β then, if these observations be true, is some 8′ too small. This would alter somewhat all the values of θ in Tables II. and III., but would produce little or no effect on the differences. The observations themselves, however, are only approximate, and can be used to give some idea of the limit of error made in giving to β the value 63° 44′ 30″, though hardly to determine its value more accurately. It is certain that β is rather too small; it is also fairly certain that it is not 10′ too small, and for our purpose at present, between these limits one value is as good as another. It therefore seemed that no real advantage would be gained by recalculating the theoretical values of θ for a value of β somewhat greater than 63° 44′ 30″.

The observations recorded above were made during the autumn of 1880, and at that time I was enabled by Lord RAYLEIGH's kindness to order, for the Cavendish Laboratory, two graduated circles, mounted so as to revolve about an axis at right angles to their planes, with tubes through their centres to carry a NICOL's prism or other such apparatus. The verniers attached read to 15″, and by means of various set screws all the adjustments requisite for accurate measurement of the position of the plane of polarization of the light emerging from the NICOL can be accomplished.

Before asking permission, therefore, to lay the preceding results before the Royal Society I waited until these circles were ready, and have repeated the experiments described above, using one of them instead of the sugar-cell to produce the rotation in the plane of polarization of the light.

The only additional adjustment necessary was to make sure that the axis of rotation of the polarizing NICOL was parallel to the incident light. The NICOL was placed at about a metre behind the slit of the collimator and adjusted by looking along the axis of the collimator until the centre of the face of the NICOL appeared to be in the axis. In the figure (7), which gives the apparatus in plan, S is the slit of the collimator, and

Fig. 7.



N the NICOL. Between the slit and the NICOL was placed a screen, A B, with another slit, P, adjusting it, so that P lay also in the axis of the collimator, and between P and S a lamp, L, so as to illuminate both slits. The lamp was so adjusted that the light from it fell along the axis of the collimator in one direction, while in the other after passing through the slit P, it fell on the face, N, of the NICOL and was there reflected. A convex lens was placed between P and N in such a position that the rays reflected from the face of N, formed an image of the slit on the screen A B. On rotating the NICOL round its axis this image moved over the screen, and if the axis of rotation is parallel to the axis of the collimator, as it should be, the centre of the image will describe a circle round P as centre. A horizontal scale was affixed to the screen along A B and the position of the NICOL adjusted until the image of the slit, as the NICOL was turned, cut this scale in points equidistant from P on either side of it.

This made it certain that the axis of the NICOL was in the same vertical plane as that of the collimator. But the axis of the collimator had been levelled with a spirit level, and by altering one of the feet on which the circle carrying the NICOL rested its axis was rendered horizontal. Thus the axis of the NICOL was rendered parallel to that of the collimator.

Behind the NICOL was placed a large lens, K, of somewhat short focal length, to act as a condenser, and at its focus the source of light, C, used in the experiment.

The lens M, the screen and lamp L, were of course removed, and when the narrow parallel beam of light from the lamp, C, after passing through the NICOL fell on the slit of the collimator, I knew that it traversed the NICOL in a direction parallel to the axis of rotation.

By means of a long handle and a HOOK'S joint the tangent screw of the circle of the polarizer could be turned by the observer when his eye was at the telescope.

The experiments were conducted in a somewhat different order to that adopted in the first set. The analyzing prism was placed so that the light fell on it at a known angle of incidence and the NICOL turned by hand until the two black bands appeared fairly close together in the ordinary spectrum. The polarizing circle was then clamped and turned by means of the handle until the bands were brought exactly into coincidence.

The reading N_0 of the NICOL circle was taken and also the deviation D_0 of the dark band in the ordinary spectrum. The NICOL circle was then turned, keeping the angle of incidence on the analyzing prism the same, until the dark bands were seen in the extraordinary spectrum and coincidence was established as before; the reading N_E of the NICOL circle was taken, and also the deviation D_E of the band in the extraordinary spectrum. The analyzing prism was then moved so as to alter the angle of incidence, and another set of observations taken, only in the reverse order. A series of values of the angles of incidence and the quantities N_0, N_E, D_0, D_E were thus observed. Each single observation was repeated five times and the mean taken and used in the calculations.

For a certain known angle of incidence, D_0 is the deviation of the light that is quenched in the ordinary spectrum. It is also however the deviation in the ordinary spectrum of the light that is missing from the extraordinary, when the coincidence between the black bands is established there.

D_0 is therefore the angle of deviation we must use in calculating the position of the plane of polarization of the incident light, when the ordinary ray only is transmitted. The experimental value of the angle determining this position, measured from some unknown zero, is N_E, for this gives us the position of the NICOL when the extraordinary wave is quenched.

Of course the theoretical values of the positions of the plane of polarization might have been determined as before from the values of the angle of incidence and ordinary refractive index, by the use of the interpolation Table I.

I wished, however, to make this series of observations entirely independent of the first, and so recalculated the interpolation table.

Since the position of the zero of the polarizer circle with reference to the analyzing prism is unknown, an unknown constant will, as before, come into our tables.

Let us determine it so that the theoretical and experimental values of the position of the plane of polarization in the first experiment on the ordinary wave agree.

Table IV. gives the values of ϕ, D_0+i, N_E, ϕ', θ_0 and the differences between θ_0 and N_E.

The last two columns give respectively the differences between the consecutive values of θ_0 and consecutive values of N_E.

The differences of θ_0 correspond to the rotation in the last column of Table II., and the differences of N_E to the constant experimental value of that rotation, viz., 4° 6′ 20″. The last two columns, therefore, enable us to compare these experiments with those recorded in Table II.

D_0+i is given, i being the angle of the prism because it occurred in this form in the formulæ, and is just as easily found at once as D_0.

Table V. gives the similar values for the case in which the extraordinary wave only is transmitted.

The zero from which N_0 is measured is of course that from which N_E has been measured.

As has already been said, each of the numbers in the columns $D+i$ and N is the mean of five observations.

The observations of the deviation rarely differed by as much as 20″, so that the mean is probably accurate to 5″. At the lower angles of incidence that is in the neighbourhood of the position of minimum deviation this would produce an error of about 1′ in the value of θ. When the angle of incidence is about 45° the error introduced into the value of θ by an error of 5″ in D is practically inappreciable.

Thus the theoretical values of θ are probably very accurate, the possible error being greatest when the angle of incidence is small, and then probably it is considerably less than 2′. The differences in the observed values of N were greater; except in the case of the first two sets recorded, when owing to the high angle of incidence very little light got through the prism and the field was very dark—the greatest difference between two observed experimental values of the same quantity was 5′. The mean error of the observations is rather under 1′ 30″, so that the recorded values of N_0 and N_E are probably accurate within this limit.

For the first two values of N_0 the extreme differences among the observations were as much as 10′, but a greater number of observations was taken. The mean error for them was 3′. In the case of the first two observations of N_E the difficulty was not so great; the extreme differences were 4′, and the mean error about 1′ 30″.

We proceed now to discuss the results of the table.

If the agreement between theory and experiment were complete, the differences recorded in the third column from the end in each table would be zero throughout.

In each case as there recorded, they increase for a time as the angle of incidence decreases and then decrease again. We of course must remember that our numbers do not give absolutely the difference between theory and experiment; there is a unknown constant to be considered which we have arbitrarily determined so as to make the difference in the first line of Table IV. zero. It may quite well be that we are

TABLE IV.—Ordinary wave only transmitted.

	ϕ.	D_0+i.	ϕ'.	θ_0-90.	$N_E-90°$.	θ_0-N_E.	Rotation theory.	Rotation experiment.
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	° ′ ″	° ′ ″
1	82 57 40	87 19 22	36 40 0	− 0 5 50	− 0 5 50	0 0	..	..
2	75 14 40	81 24 40	35 35 0	2 45 40	2 31 15	15 25	2 51 30	2 37 5
3	67 12 55	76 32 50	33 41 30	7 6 40	6 43 50	22 50	4 21 0	4 12 35
4	59 59 55	73 11 14	31 24 0	12 0 10	11 27 50	32 20	4 53 30	4 44 0
5	54 43 10	71 17 26	29 24 40	15 44 30	15 13 0	31 30	3 44 20	3 45 10
6	49 5 40	69 43 36	27 3 0	20 0 10	19 30 35	29 35	4 15 40	4 17 35
7	43 46 55	68 43 10	24 35 50	24 6 30	23 35 50	30 40	4 6 20	4 5 15
8	38 41 40	68 9 0	22 5 20	27 58 20	27 32 50	25 30	3 51 50	3 57 0
9	32 51 0	67 58 42	19 2 30	32 14 30	31 54 30	20 0	4 16 10	4 21 40
10	27 38 10	68 19 54	16 12 0	35 49 30	35 31 20	18 10	3 35 0	3 36 50
11	21 17 30	68 34 28	12 36 50	39 50 20	39 36 30	13 50	4 0 50	4 5 10

TABLE V.—Extraordinary wave only transmitted.

	ϕ.	D_E+i.	ϕ''.	θ_E.	N_0.	$N_0-\theta_E$.	Rotation theory.	Rotation experiment.
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	′ ″	° ′ ″	° ′ ″
1	82 57 40	85 31 20	37 42 20	0 48 30	0 37 0	−11 30	..	..
2	75 14 40	79 38 20	36 33 20	2 10 15	2 8 50	− 1 25	1 21 45	1 31 50
3	67 12 55	74 47 28	34 37 40	5 11 15	5 24 50	13 35	3 1 0	3 16 0
4	59 59 55	71 25 8	32 16 30	9 15 15	9 34 50	19 35	4 4 0	4 10 0
5	54 43 10	69 28 57	30 14 40	12 58 0	13 17 45	19 45	3 42 45	3 42 55
6	49 5 40	67 51 28	27 50 50	17 17 0	17 38 6	21 6	4 19 0	4 20 21
7	43 46 55	66 43 56	25 21 10	21 38 30	31 58 20	19 30	4 21 30	4 20 14
8	38 41 40	65 59 43	22 46 0	25 55 30	26 12 10	16 40	4 17 0	4 13 50
9	32 51 0	65 33 18	19 41 0	30 39 0	30 49 50	10 50	4 43 30	4 37 40
10	27 38 10	65 33 18	16 47 20	34 38 15	34 45 30	7 15	3 59 15	3 55 40
11	21 17 30	66 9 22	13 7 20	39 6 30	39 12 0	5 30	4 28 15	4 26 30

wrong in so doing. The table, however, shows us conclusively that we cannot so determine this constant as to bring theory and experiment into exact agreement.

The last two columns in each table enable us to compare these results with those of our former experiments.

The one gives us the rotation of the plane of polarization when only one wave traversed the crystal according to the electro-magnetic theory, the other the measured value of the same rotation.

Consider Table IV. first. At high angles of incidence the theoretical value of rotation is greater than the experimental. As the angle of incidence decreases the two become more nearly equal and agree within the limits of the error of experiment between angles of incidence ranging from 55° to 45°. As the angle of incidence still further decreases the theoretical value of the rotation becomes decidedly less than the experimental.

In the main this agrees with Table II. At high angles of incidence the rotation given by theory is greatly in excess of that given by experiment, the two agree between 55° and 40°, but for the lower angles of incidence Table II. shows a slight increase in the theoretical value as compared with the experimental.

Turning now to the last two columns in Table V., we see that the theoretical rotation is at high angles of incidence less than the experimental, that as the angle of incidence decreases the two tend to become equal and agree very closely for angles of incidence between 55° and 45°, while from that point onwards the theoretical rotation is bigger than the experimental. The change in the relative values of the two is very regular, while the actual change of sign in their difference occurs between the values 49° 5′ 40″ and 43° 46′ 45″ of the angle of incidence.

Referring to Table III. we see that this is exactly the state of affairs there indicated. The theoretical rotation is at first less than the experimental and increases gradually as the angle of incidence decreases, becoming the greater between the values 50° 27′ 30″ and 45° 15′, almost the same limits as above, of the angle of incidence, and continuing so throughout the rest of the arc examined.

With the exception then of one or two observations recorded at the end of Table II., the results of the two series of experiments, the one made during the autumn of 1880, the other during the summer of 1881, agree and lead us to the conclusion that the laws arrived at by the electro-magnetic theory connecting the planes of polarization of the incident and refracted rays in the case of refraction at the surface of a uniaxal crystal are not exact but are probably close approximations to the truth.

These same laws have been arrived at by NEUMANN,* MACCULLAGH,† KIRCHKOFF ‡ and others on various assumptions as to the nature of the ether; and in the case of an

* Abhand. der Akad. der Berlin, 1835.

† Irish Trans., 1837.

‡ Abhand. der Akad. der Berlin, 1876.

isotropic refracting media agree with FRESNEL's formulæ for the relative positions of the planes of polarization of the incident and reflected rays, though not for the refracted.

The experiments having all been conducted with the same piece of spar do not afford data for deciding whether the differences observed are functions only of the angle of incidence, or whether, as is more likely, they depend partly on it and partly on the position of the face of incidence with reference to the axis of the crystal. Though in no case large, they are certainly considerably greater than the possible errors of experiment.

The experiments have been conducted by Lord RAYLEIGH's kind permission in one of the rooms of the Cavendish Laboratory, Cambridge, and the apparatus used is chiefly the property of the Laboratory.

In conclusion, I would refer to a paper on the same subject by F. E. NEUMANN ("Beobactungen über den Einfluss der Krystaleflächen auf das reflectirite Licht." POGG. Ann., xlii.).

He polarized light by passing it through a tourmaline, and then allowed it to fall on a prism of Iceland spar at a known angle of incidence, observing the position of the plane of polarization when only one ray passed through the crystal. The light of a lamp was used and the crystal prism achromatised approximately for the ordinary ray. The prism was capable of rotation about a normal to the face on which the light fell so as to alter relatively to the axis of the crystal the position of the plane of incidence. The theoretical values of the position of the plane of polarization were calculated from formulæ given by NEUMANN (Abhandlungen der Akad. der Berlin, 1835), and MACCULLAGH (Irish Transactions, 1837), which are identical with those deduced from the electro-magnetic theory. In the eight observations recorded, the differences between the calculated and observed values of the azimuth of the plane of polarization range from $-8'$ to $+4'$.

Each observed value is the mean of 40 observations, but it is not stated how far the observations differ among themselves; observations were only made for a small number of values of ϕ, the angle of incidence, two at most for each position of the plane of incidence relatively to the axis of the crystal, and the values of ϕ chosen were 40°, 45°, 50° and 55°.

Now, it will be seen on referring to my tables, that for these values of ϕ the differences there recorded are small. Nothing is said in NEUMANN's paper as to the method adopted to determine the values of ϕ' and ϕ'', the angles of refraction of the ordinary and extraordinary rays. In fact, they are both indeterminate, seeing that the experiment was made with white light. These are the only other experiments made to test the theory so far as it concerns the refraction of light.

The same paper of NEUMANN's contains a series of experiments on the position of the plane of polarization of light reflected from the surface of the crystal under various circumstances, the results of which are compared with theory.

The comparison is also extended to some earlier experiments of SEEBECK'S (POGG. Ann. xxi., xxii., and xxxviii.). As far as one can judge from the details given, the differences are hardly greater than the possible errors of the experiment, for in all cases lamp light was used and the position of the plane of polarization determined by the quenching of one of the rays in a double image prism, and both these circumstances preclude great accuracy. On the other hand, the number of individual observations was very large, so that the mean deserves considerable weight.